QUESTION ONE *(3 Marks)* **START A NEW BOOKLET**

MULTIPLE CHOICE: Write the correct alternative on your writing paper.

1. Which of the following gives $|z| = 2\sqrt{5}$ and $\arg z = -\pi + \tan^{-1} 2$ **1**

(A) $2+4i$

(B) $-2+4i$

(C) $-2-4i$

(D) $2-4i$

2. Which of the following represent the three cubed roots of -8 **1**

(A) $-2,\ 1-\sqrt{3}i,\ 1+\sqrt{3}i$

(B) $-2,\ -1-\sqrt{3}i,\ 1+\sqrt{3}i$

(C) $-2,\ 1-\sqrt{3}i,\ 1-\sqrt{3}i$

(D) $2,\ 1-\sqrt{3}i,\ 1+\sqrt{3}i$

3. If z and w are the roots of the equation $3x^2+(2-i)x+(4+i)=0$, **1**
which of the following represent $\dfrac{1}{\overline{z}}+\dfrac{1}{\overline{w}}$

(A) $\dfrac{-7-6i}{17}$

(B) $\dfrac{-7+6i}{15}$

(C) $\dfrac{-7-6i}{15}$

(D) $\dfrac{7+6i}{17}$

QUESTION TWO *(23Marks)*

(a) Let $z = 3 + i$ and $w = 2 - 5i$. Find in the form $x + iy$,

(i) z^2 **1**

(ii) $\overline{z} + w$ **1**

(iii) $\dfrac{w}{z}$ **1**

(b) Find all pairs of integers x and y that satisfy $(x + iy)^2 = 24 + 10i$ **3**

(c) Sketch the region in the complex plane where the inequalities **3**

$$|z - 1 - i| \le 2 \text{ and } 0 < \arg(z - 1 - i) < \frac{\pi}{4}$$

hold simultaneously.

(d) (i) Write $1 + i$ in the form $r(\cos\theta + i\sin\theta)$ **2**

(ii) Hence, or otherwise , find $(1 + i)^{17}$ in the form $x + iy$, **3**
where a and b are integers.

(e) Prove by Mathematical Induction that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ **4**
for all integers $n \ge 1$.

(e) If ω is the non real cube root of 1,

(i) Prove that $(1 + 2\omega + 3\omega^2)(1 + 3\omega + 2\omega^2) = 3$. **2**

(ii) Prove that . $(1 + 2\omega + 3\omega^2) + (1 + 3\omega + 2\omega^2) = -3$ **1**

(iii) Hence find the exact values of $(1 + 2\omega + 3\omega^2)$ *and* $(1 + 3\omega + 2\omega^2)$. **2**

QUESTION THREE *(28 Marks)* **START A NEW BOOKLET**

(a) Draw separate sketches of the following, clearly indicating any critical points. asymptotes, discontinuities etc. Make each sketch about half a page.

(i) $y=(x+1)(x-1)$ **1**

(ii) $y=\dfrac{1}{(x+1)(x-1)}$ **2**

(iii) $y=\sqrt{(x+1)(x-1)}$ **2**

(iv) $y^2=(x+1)(x-1)$ **2**

(v) $y=\left|(x+1)(x-1)\right|$ **2**

(vi) $y=\left[(x+1)(x-1)\right]^2$ **2**

(vii) $|y|=(x+1)(x-1)$ **2**

(viii) $y=\log_e(x+1)(x-1)$ **2**

(b) For the function $f(x)=\dfrac{x^4}{x^2-1}$.

(i) Determine whether $f(x)$ is odd, even or neither. **1**

(ii) Find the coordinates of the stationary points. **3**

(iii) By considering large values of $|x|$ and any discontinuities, sketch the graph of $f(x)=\dfrac{x^4}{x^2-1}$ showing all essential features. **4**

(c) The equation of a curve is given by $3x^2+y^2-2xy-8x+2=0$.

(i) Find $\dfrac{dy}{dx}$. **2**

(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y=2x$. **3**

<u>QUESTION FOUR</u> *(3 Marks)* **START A NEW BOOKLET**

MULTIPLE CHOICE: Write the correct alternative on your writing paper

1. When $P(x) = x^4 - 1$ is factorised over the complex field it may be written as

   (A) $P(x) = (x^2 - 1)(x^2 + 1)$
   (B) $P(x) = (x-1)(x+1)(x^2 + 1)$
   (C) $P(x) = (x-1)(x+1)(x+i)(x-i)$
   (D) $P(x) = (x^2 - 1)(x-i)^2$

2. If α, β, γ are the roots of the equation $x^3 - 8x^2 - 4x + 12 = 0$, then the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is

   (A) $\frac{8}{3}$
   (B) $1 + \frac{1}{-8} + \frac{1}{-12}$
   (C) $\frac{1}{3}$
   (D) $\frac{1}{-12}$

3. Given $\frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$, the values of A and B respectively are

   (A) 1, -1
   (B) -1,1
   (C) $\frac{1}{4}, -\frac{1}{4}$
   (D) $-\frac{1}{4}, \frac{1}{4}$

QUESTION FIVE *(23Marks)* **START A NEW BOOKLET**

(a) The equation $x^3 - 4x^2 + 5x + 2 = 0$ has roots α, β *and* γ.

Find (i) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **3**

(b) Factorise $3x^4 + 17x^3 + 30x^2 + 12x - 8 = 0$ completely if it has a root of multiplicity 3. **3**

(c) (i) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

(ii) Find the roots of the equation $8x^3 - 6x - 1 = 0$ in terms of $\cos\theta$. **2**

(iii) Hence evaluate $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$. **2**

(d) If $\frac{2x+31}{(x+1)^3(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{(x+2)}$ **4**

find the values of A, B, C and D.

(e) The roots of $x^3 + 3px + q = 0$ are α, β *and* γ (none of which are equal to 0).

(i) Find the monic equation with roots $\frac{\beta\gamma}{\alpha}, \frac{\alpha\gamma}{\beta}$ *and* $\frac{\alpha\beta}{\gamma}$, giving the coefficients in terms of p and q. **3**

(ii) Deduce that if $\gamma = \alpha\beta$ then $(3p - q)^2 + q = 0$. **2**

END OF PAPER

Year 12 Extension 2 Mini Examination 2012

Question 1

1. $|z| = 2\sqrt{5}$ $\quad \arg z = -\pi + \tan^{-1} 2$

z lies in the 3rd quadrant

$\therefore z = -2 - 4i$ $\qquad$ (C)

2. Let $z^3 = -8$

$$z^3 + 8 = 0$$

$$(z+2)(z^2 - 2z + 4) = 0$$

$$(z+2)\left[(z-1)^2 + 3\right] = 0$$

$$(z+2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i) = 0$$

$\therefore z = -2,\ 1-\sqrt{3}i,\ 1+\sqrt{3}i$ $\qquad$ (A)

3. $3x^2 + (2-i)x + (4+i) = 0$

$$\alpha + \beta = \frac{-b}{a} = \frac{-2+i}{3}$$

$$\alpha\beta = \frac{c}{a} = \frac{4+i}{3}$$

$$\frac{1}{z} + \frac{1}{w} = \frac{w+z}{wz} = \frac{\frac{-2+i}{3}}{\frac{4+i}{3}} = \frac{-2+i}{4+i} = \frac{-2+i}{4+i} \times \frac{4-i}{4-i} = \frac{-8+6i-i^2}{17} = \frac{-7+6i}{17}$$

$\therefore \frac{1}{\bar{z}} + \frac{1}{\bar{w}} = \frac{-7-6i}{17}$ $\qquad$ (A)

Question 2

a) $z = 3+i$ $\quad w = 2-5i$

(i) $z^2 = (3+i)^2 = 9 + 6i + i^2 = 8 + 6i$

(ii) $\bar{z} + w = 3 - i + 2 - 5i = 5 - 6i$

(iii) $\frac{w}{z} = \frac{2-5i}{3+i} = \frac{2-5i}{3+i} \times \frac{3-i}{3-i} = \frac{6-17i+5i^2}{10} = \frac{1}{10} - \frac{17}{10}i$

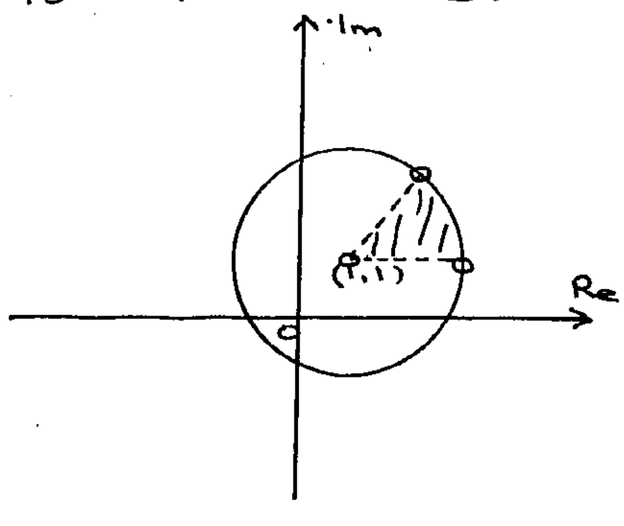
b) $(x+iy)^2 = 24 + 10i$

$$x^2 - y^2 = 24$$

$$xy = 5$$

$\therefore$ The 2 square roots are $\pm(5+i)$

c) $|z-1-i| \le 2$ $\quad 0 < \arg(z-1-i) < \frac{\pi}{4}$



d) (i) $z = 1+i$

$|z| = \sqrt{1^2+1^2}$ $\quad$ $\arg z = \tan^{-1} 1 \quad -\pi < \theta \leq \pi$

$= \sqrt{2}$ $\quad$ $= \frac{\pi}{4}$

$\therefore 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$(1+i)^{17} = (\sqrt{2})^{17}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$

$= 256\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 256 + 256i$

e) Show $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \quad n \geq 1$

Step 1 Test $n=1$

$L.H.S = (\cos\theta + i\sin\theta)^1$

$= \cos\theta + i\sin\theta$

$R.H.S = \cos\theta + i\sin\theta$

$L.H.S = R.H.S$

$\therefore$ Result is true for $n=1$

Step 2. Assume that the result is true for $n=k$

that is $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

Step 3 Hence show the result is true for $n=k+1$

that is $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

$L.H.S = (\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ by step 2

$= \cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta + i^2\sin\theta\sin k\theta$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

$= R.H.S$

Step 4 Since the result is true for $n=1$ then from Step 3 it is true for $n=1+1=2$, and then for $n=3$, and so on by the process of mathematical induction it is true for all positive integral values of n

e) $\omega^3 = 1$

$\omega^3 - 1 = 0$

$(\omega-1)(\omega^2+\omega+1) = 0$

$\omega - 1 \neq 0 \quad \therefore \omega^2+\omega+1 = 0$

(i) $(1+2\omega+3\omega^2)(1+3\omega+2\omega^2)$

$= 1+3\omega+2\omega^2+2\omega+6\omega^2+4\omega^3+3\omega^2+9\omega^3+6\omega^4$

$= 1+5\omega+11\omega^2+13\omega^3+6\omega^4$

$= 1+5\omega+11\omega^2+13+6\omega$

$= 14+11\omega+11\omega^2$

$= 14+11(\omega^2+\omega)$

$= 14-11$

$= 3$

NB ① $\omega^3 = 1$

② $\omega^4 = \omega$

③ $\omega^2+\omega+1 = 0$

$\omega^2+\omega = -1$

(ii) $1+2\omega+3\omega^2+1+3\omega+2\omega^2 = 2+5\omega+5\omega^2$

$= 2+5(\omega+\omega^2)$

$= 2-5$

$= -3$

(iii) From parts (i) and (ii) $(1+2\omega+3\omega^2)$ and $(1+3\omega+2\omega^2)$ are roots of $x^2+3x+3=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{3^2-4\times1\times3}}{2\times1}$$

$$= \frac{-3 \pm i\sqrt{3}}{2}$$

$\therefore 1+2\omega+3\omega^2 = \dfrac{-3-i\sqrt{3}}{2}$

$1+3\omega+2\omega^2 = \dfrac{-3+i\sqrt{3}}{2}$

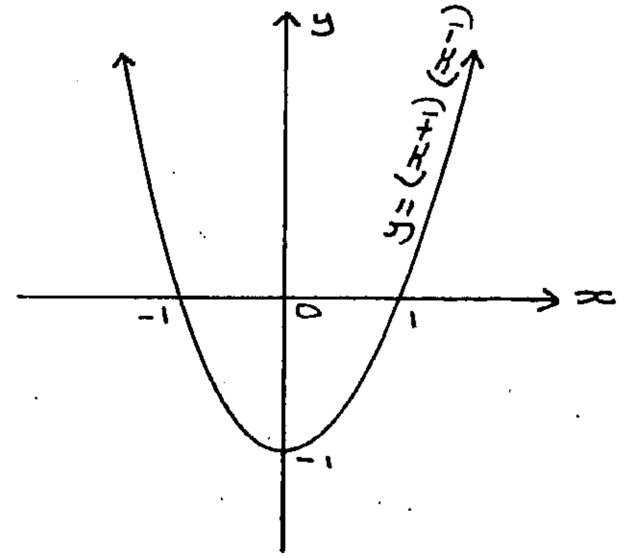
NB $\operatorname{Im}(\omega) = -\operatorname{Im}(\omega^2)$

$\operatorname{Im}(2\omega+3\omega^2) = \operatorname{Im}(\omega^2) < 0$

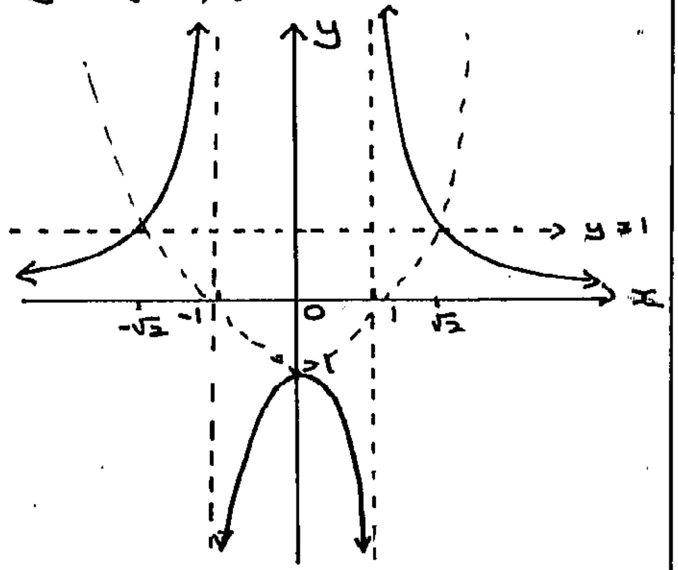
$\operatorname{Im}(3\omega+2\omega^2) = \operatorname{Im}\omega > 0$

Question 3

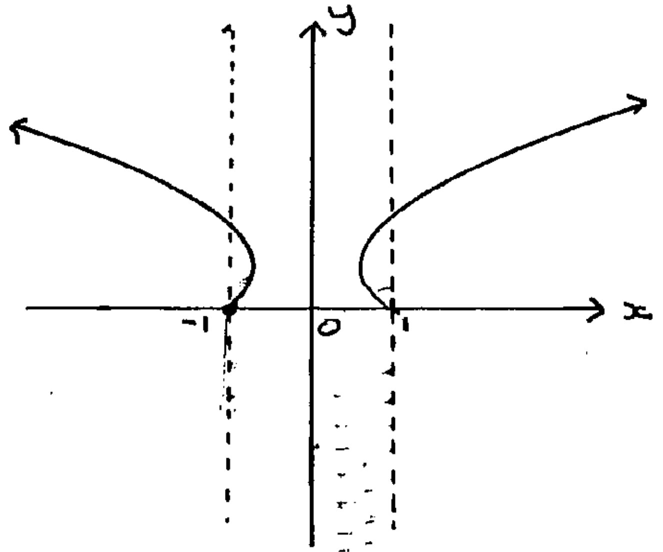
a) (i) $y = (x+1)(x-1)$



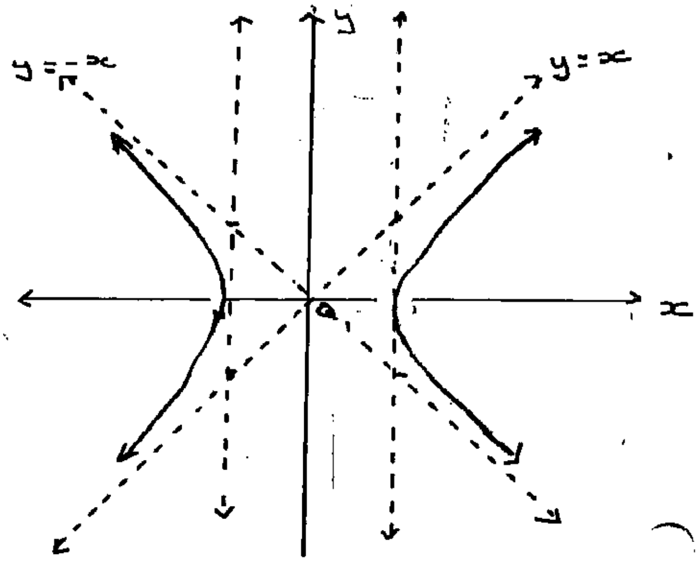
(ii) $y = \dfrac{1}{(x-1)(x+1)}$



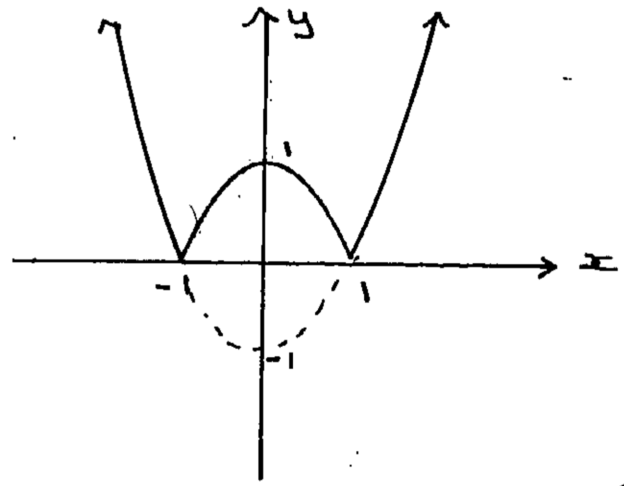
(iii) $y = \sqrt{(x-1)(x+1)}$



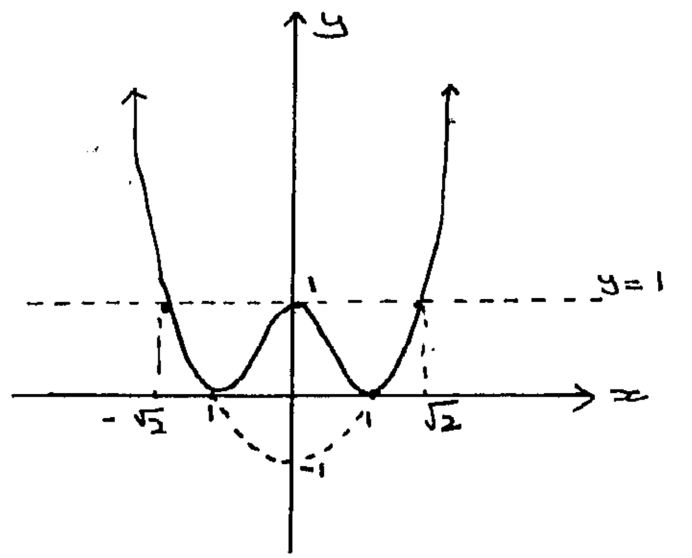
(iv) $y^2 = (x+1)(x-1)$



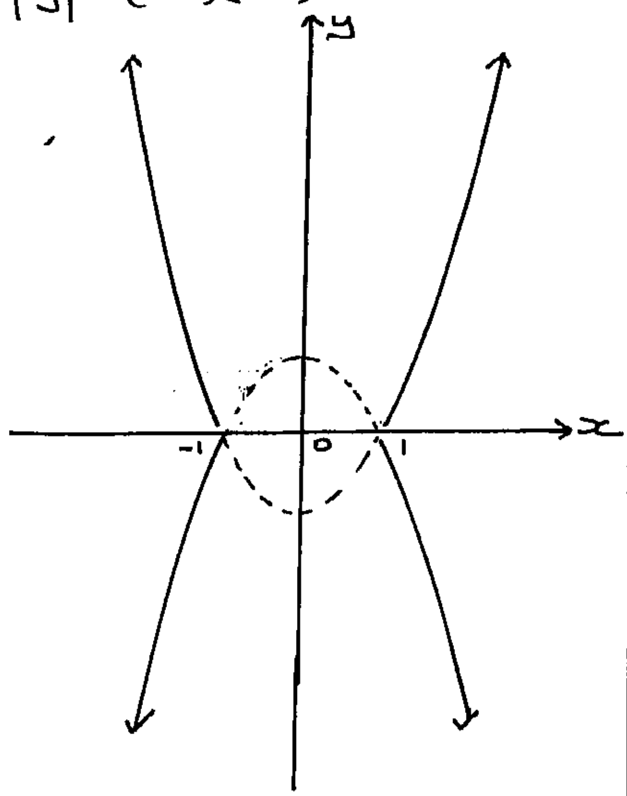
(v) $y = |(x-1)(x+1)|$



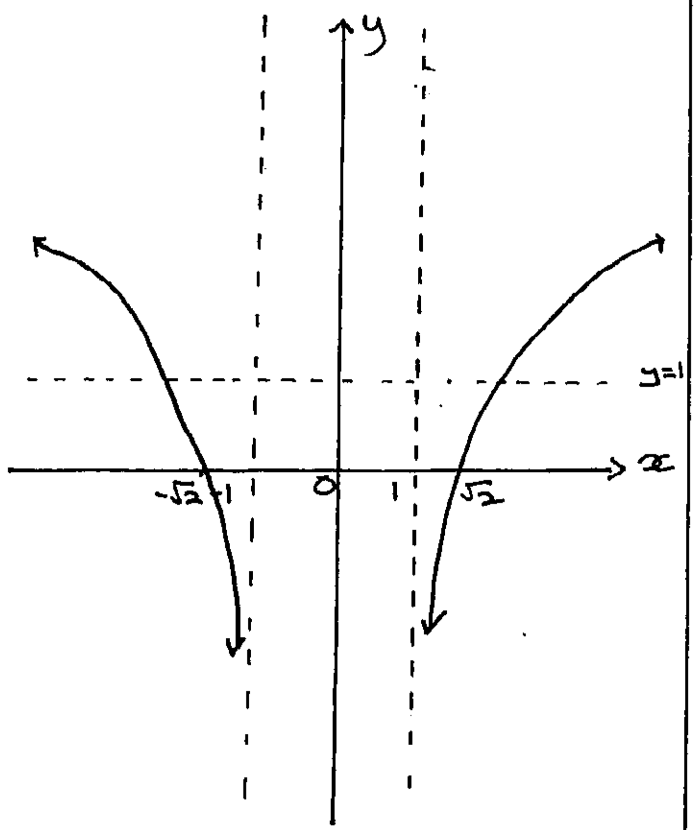
(vi) $y = [(x+1)(x-1)]^2$



vii) $|y| = (x-1)(x+1)$



viii) $y = \log_e (x+1)(x-1)$



b) $f(x) = \dfrac{x^4}{x^2-1}$

$f(-x) = \dfrac{(-x)^4}{(-x)^2-1}$

$= \dfrac{x^4}{x^2-1}$

$\therefore f(x) = f(-x)$

$\therefore f(x) = \dfrac{x^4}{x^2-1}$ is an even function.

ii) $f(x) = \dfrac{x^4}{x^2-1}$

$u = x^4 \qquad v = x^2-1$

$\dfrac{du}{dx} = 4x^3 \qquad \dfrac{dv}{dx} = 2x$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2-1)4x^3 - x^4 \times 2x}{(x^2-1)^2}$$

$$= \frac{2x^3(2x^2-2-x^2)}{(x^2-1)^2}$$

$$= \frac{2x^3(x^2-2)}{(x^2-1)^2}$$

$$= \frac{2x^3(x-\sqrt{2})(x+\sqrt{2})}{(x^2-1)^2}$$

For stationary points

$f'(x) = 0$

$\therefore 2x^3(x-\sqrt{2})(x+\sqrt{2}) = 0$

$\therefore x = 0, \pm\sqrt{2}$

$$f(x) = \frac{x^4}{x^2-1}$$

$$f(0) = 0$$

$$f(\sqrt{2}) = \frac{(\sqrt{2})^4}{(\sqrt{2})^2-1}$$

$$= 4$$

$$f(-\sqrt{2}) = \frac{(-\sqrt{2})^4}{(-\sqrt{2})^2-1}$$

$$= 4$$

$\therefore$ The stationary points are $(0,0)$ $(\sqrt{2},4)$ and $(-\sqrt{2},4)$

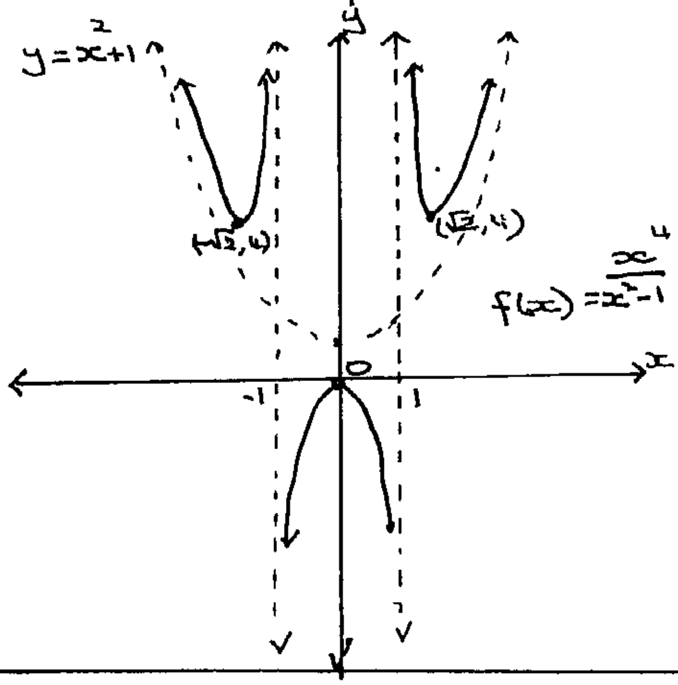
(iii) vertical asymptotes are $x = \pm 1$

$$f(x) = \frac{x^4}{x^2-1}$$

$$= x^2+1 + \frac{1}{x^2-1}$$

as $x \longrightarrow \infty$ $f(x) \longrightarrow x^2+1$

as $x \longrightarrow -\infty$ $f(x) \longrightarrow x^2+1$



c) $3x^2+y^2-2xy-8x+2=0$ —①

$$6x+2y\frac{dy}{dx}-\left(2x\frac{dy}{dx}+2y\right)-8=0$$

$$(2y-2x)\frac{dy}{dx} = 8+2y-6x$$

$$\frac{dy}{dx} = \frac{8+2y-6x}{2y-2x}$$

$$\therefore \frac{dy}{dx} = \frac{4+y-3x}{y-x}$$

(ii) As the tangent is parallel to $y=2x$, $\frac{dy}{dx}=2$

$$\frac{4+y-3x}{y-x} = 2$$

$$4+y-3x = 2y-2x$$

$$y = 4-x \quad \text{—②}$$

sub ② in ①

$$3x^2+y^2-2xy-8x+2=0$$

$$3x^2+(4-x)^2-2x(4-x)-8x+2=0$$

$$3x^2+16-8x+x^2-8x+2x^2-8x+2=0$$

$$6x^2-24x+18=0$$

$$x^2-4x+3=0$$

$$(x-3)(x-1)=0$$

$$x=3,1$$

when $x=3$

sub ② $y = 4-x$
$= 4-3$
$= 1$

when $x=1$

sub ② $y = 4-x$
$= 4-1$
$= 3$

$\therefore$ The points are $(3,1)$ $(1,3)$

Question 4

1. $P(x) = x^4 - 1$

$= (x^2-1)(x^2+1)$

$= (x-1)(x+1)(x-i)(x+i)$

(C)

2. $x^3 - 8x^2 - 4x + 12 = 0$

$\alpha+\beta+\gamma = \frac{-b}{a}$

$= 8$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$= -4$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -12$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$= \frac{-4}{-12}$

$= \frac{1}{3}$ (C)

3. $\frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$1 = A(x-3) + B(x+1)$

sub $x=3$

$1 = B(3+1)$

$B = \frac{1}{4}$

sub $x=-1$

$1 = A(-1-3)$

$A = -\frac{1}{4}$

$\therefore A = -\frac{1}{4},\ B = \frac{1}{4}$ (D)

Question 5

a) $x^3 - 4x^2 + 5x + 2 = 0$

$\alpha+\beta+\gamma = \frac{-b}{a}$

$= 4$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$= 5$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -2$

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 4^2 - 2\times5$

$= 6$

(ii) $\alpha^3 = 4\alpha^2 - 5\alpha - 2$ —— ①

$\beta^3 = 4\beta^2 - 5\beta - 2$ —— ②

$\gamma^2 = 4\gamma^2 - 5\gamma - 2$ —— ③

① + ② + ③

$\alpha^3+\beta^3+\gamma^3 = 4(\alpha^2+\beta^2+\gamma^2) - 5(\alpha+\beta+\gamma) - 2\times3$

$= 4(6) - 5(4) - 6$

$= -2$

b) $3x^4 + 17x^3 + 30x^2 + 12x - 8 = 0$

$P(x) = 3x^4 + 17x^3 + 30x^2 + 12x - 8$

$P'(x) = 12x^3 + 51x^2 + 60x + 12$

$P''(x) = 36x^2 + 102x + 60$

$= 2(18x^2 + 51x + 30)$

$= 2(x+2)(18x+15)$

$P(-2) = 0$

$P'(-2) = 0$

$P''(-2) = 0$

$\therefore$ $(x+2)$ is a triple root

$\therefore 3x^4 + 17x^3 + 30x^2 + 12x - 8 = 0$

$(x+2)^3(3x-1) = 0$

by inspection

c) By De Moivre's Theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$$\begin{aligned}
\text{L.H.S} &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta(i\sin\theta)^2 + i^3\sin^3\theta \\
&= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

Equating real parts

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
&= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta
\end{aligned}$$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

ii) $8x^3 - 6x - 1 = 0$

$4x^3 - 3x = \frac{1}{2}$

But $\cos 3\theta = \frac{1}{2}$

NB $3\theta = \pm\frac{\pi}{3} + 2n\pi$, n is integral

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$\therefore$ The roots of $8x^3 - 6x - 1 = 0$ are $\cos\frac{\pi}{9}$, $\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$ and $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$

(iii) $8x^3 - 6x - 1 = 0$

product of the roots $= \frac{-d}{a}$

$= \frac{1}{8}$

$\therefore \cos\frac{\pi}{9} \times -\cos\frac{4\pi}{9} \times -\cos\frac{2\pi}{9} = \frac{1}{8}$

$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$

d) $$\frac{2x+31}{(x-1)^3(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+2)}$$

$$2x+31 = A(x-1)^2(x+2) + B(x-1)(x+2) + C(x+2) + D(x-1)^3$$

Let $x=1$ $\quad 2(1)+31 = C(1+2)$

$$33 = 3C$$
$$C = 11$$

Let $x=-2$ $\quad 2(-2)+31 = D(-2-1)^3$

$$27 = -27D$$
$$D = -1$$

Let $x=0$ $\quad 2(0)+31 = A(-1)^2(2) + B(-1)(2) + 11(2) - 1(-1)^3$

$$31 = 2A - 2B + 22 + 1$$
$$8 = 2A - 2B$$
$$4 = A - B \quad \text{——} \; (1)$$

Let $x=-1$ $\quad 2(-1)+31 = A(-1-1)^2 + B(-1-1)(-1+2) + 11(-1+2) - 1(-1-1)^3$

$$29 = 4A - 2B + 11 + 8$$
$$10 = 4A - 2B$$
$$5 = 2A - B \quad \text{——} \; (2)$$

Solve (1) and (2) simultaneously

$$A - B = 4 \quad \text{——} \; (1)$$
$$2A - B = 5 \quad \text{——} \; (2)$$

(1) − (2) $\quad -A = -1$

$$A = 1$$

sub (1) $\quad A - B = 4$

$$1 - B = 4$$
$$B = -3$$

$$\therefore A = 1, \quad B = -3, \quad C = 11, \quad D = -1$$

e) $x^3 + 3px + q = 0$

$\alpha + \beta + \gamma = \frac{-b}{a}$

$= 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= 3p$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -q$

sum of the roots $= \frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\alpha\gamma}{\beta}$

$= \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2}{\alpha\beta\gamma}$

$= \frac{(\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)}{\alpha\beta\gamma}$

$= \frac{(3p)^2 - q(0)}{-q}$

$= \frac{9p^2}{-q}$

sum of the roots 2 at a time $= \frac{\alpha\beta}{\gamma} \times \frac{\beta\gamma}{\alpha} + \frac{\beta\gamma}{\alpha} \times \frac{\gamma\alpha}{\beta} + \frac{\gamma\alpha}{\beta} \times \frac{\alpha\beta}{\gamma}$

$= \beta^2 + \gamma^2 + \alpha^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0 - 2(3p)$

$= -6p$

Product of the roots $= \frac{\alpha\beta}{\gamma} \times \frac{\alpha\gamma}{\beta} \times \frac{\beta\gamma}{\alpha}$

$= -q$

$\therefore$ The monic equation is $x^3 + \frac{9p^2}{q}x^2 - 6px + q = 0$

(ii) If $\gamma = \alpha\beta$ then one of the roots, $\frac{\alpha\beta}{\gamma} = \frac{\gamma}{\gamma} = 1$

$$\therefore \quad x^3 + \frac{9p^2}{q}x^2 - 6px + q = 0$$

$$1 + \frac{9p^2}{q} - 6p + q = 0$$

$$q + 9p^2 - 6pq + q^2 = 0$$

$$q + (3p-q)(3p-q) = 0$$

$$\therefore \quad (3p-q)^2 + q = 0$$